5.2 Exploring Quotients of Polynomial Functions (Rational Functions)

A Rational Functions A *rational function* is a function of the form: $$f(x)=\frac{P(x)}{Q(x)}$$ where $P(x)$ and $Q(x)$ are *polynomial functions*.	Ex 1. Verify if the following functions are or are not rational functions. a) $f(x)=\frac{x^2-x+2}{x^2-1}$ b) $f(x)=\frac{x}{\sqrt{x}+x^2}$ c) $f(x)=x^{-2}+\frac{1}{x}$
B Domain The *domain* of a rational function is determined by the restriction $$Q(x)\neq 0$$	Ex 2. Find the domain of each rational function. a) $f(x)=\frac{x^2-1}{x-1}$ b) $f(x)=\frac{x+1}{x^2-4}$ c) $f(x)=\frac{x^2-x}{6x^3+x^2-2x}$
C y-intercept Point The *y-intercept point* for any function $y=f(x)$ is the point $(0,f(0))$ if 0 is in the domain of the function f.	Ex 3. Find the y-intercept of each rational function. a) $f(x)=\frac{x^2+1}{x}$ b) $f(x)=\frac{x+3}{x^2-1}$
D Holes The rational function $f(x)=P(x)/Q(x)$ has a *hole* in its graph at $x=a$ if $$P(a)=0 \text{ and } Q(a)=0$$ and if the *simplified formula* of function $f(x)$ is *defined* at $x=a$.	Ex 4. Sketch the graph of the following rational functions. a) $f(x)=\frac{x^2}{x}$ b) $f(x)=\frac{x^2}{x^4}$ c) $f(x)=\frac{x^2-4}{x+2}$ d) $f(x)=\frac{x^3-1}{x-1}$
E Zeros The rational function $f(x)=P(x)/Q(x)$ has a *zero* at $x=a$ if $$P(a)=0 \text{ and } Q(a)\neq 0$$	Ex 5. Find the zeros of the following rational functions. a) $f(x)=\frac{x-1}{x^2-1}$ b) $f(x)=\frac{x^2-1}{x+1}$ c) $f(x)=\frac{x^2+1}{x-1}$

F Vertical Asymptotes

The vertical line $x = a$ is a *vertical asymptote* for the graph of the function $f(x)$ if the *value* of the function becomes *unbounded* ($y = f(x) \to \pm\infty$) as x approaches a from the left or from the right.

Note. If $x = a$ is a vertical asymptote for a rational function $f(x) = P(x)/Q(x)$ then (*after simplification*)

$$P(a) \neq 0 \text{ and } Q(a) = 0$$

Note. A vertical asymptote *splits* the graph of a function in *branches*.

Ex 6. For each case, find the equation of the vertical asymptotes.

a) $f(x) = \dfrac{x}{x^2 + 1}$

b) $f(x) = \dfrac{x+3}{x^2 - x - 6}$

c) $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$

G Horizontal Asymptotes

The horizontal line $y = c$ is a *horizontal asymptote* for the graph of the function $f(x)$ if $y = f(x) \to a$ as $|x|$ becomes *unbounded* ($x \to \pm\infty$).

Note. Some functions may have two different horizontal asymptotes (one as $x \to \infty$ and one as $x \to -\infty$).

Note. Rational functions may have *at most one* horizontal asymptote.

In the case of a rational function:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + ..a_1 x + a_0}{b_m x^m + ..b_1 x + b_0}$$

- If $P(x)$ and $Q(x)$ have the *same degree* ($n = m$) then the equation of the horizontal asymptote is $y = \dfrac{a_n}{b_m}$.
- If the degree of $P(x)$ is *less* that the degree of $Q(x)$ then the equation of the horizontal asymptote is $y = 0$.
- If the degree of $P(x)$ is *greater* that the degree of $Q(x)$ then the rational function *does not have* a horizontal asymptote.

Ex 7. For each case, find the equation of the horizontal asymptote (if exists).

a) $f(x) = \dfrac{x^2 - 2x + 1}{3x^3 - 3x + 5}$

b) $f(x) = \dfrac{2x^2 + 3}{x + 1}$

c) $f(x) = -\dfrac{x^3 + x - 1}{(2x-1)(x+1)^2}$

H Graph Sketching

Use the x-intercepts, y-intercept, symmetry, vertical and horizontal asymptote to sketch the graph of a rational function.

Ex 8. Sketch the graph for the following rational functions.

a) $f(x) = \dfrac{x - 1}{x^2 - 1}$

b) $f(x) = \dfrac{x^2 - 1}{x^2 - 4}$

Reading: Nelson Textbook, Pages 258-261

Homework: Nelson Textbook, Page 262: #1, 2, 3

